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DIMENSION FREE ESTIMATES FOR THE OSCILLATION OF RIESZ TRANSFORMS*

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ABSTRACT

In this paper we establish dimension free $L^p(\mathbb{R}^n, |x|^\alpha)$ norm inequalities $(1 < p < \infty)$ for the oscillation and variation of the Riesz transforms in $\mathbb{R}^n$. In doing so we find A_p-weighted norm inequalities for the oscillation and the variation of the Hilbert transform in $\mathbb{R}$. Some weighted transference results are also proved.

Introduction

Throughout $(X, \mathcal{F}, \mu)$ will denote an arbitrary σ-finite measure space. Let $\{T_r\}$ be a family of operators bounded from $L^p(X, \mathcal{F}, \mu)$ into itself for some p in the range $1 < p < \infty$ and such that the limit $Tf = \lim_{r \searrow 0} T_r f$, for functions $f \in L^p(X, \mathcal{F}, \mu)$, exists in some sense. A classical method of measuring the speed of convergence of the family $\{T_r\}$ is to consider "square functions" of the type $(\sum_{i=1}^{\infty} |T_{r_i} f - T_{r_{i+1}} f|^2)^{1/2}$ where $r_i \searrow 0$. In the last decade and mainly

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in the context of ergodic theory (see [JKRW] and the references there), other expressions have been considered to measure the speed of convergence.

Let T be an operator such that $T = \lim_{r \searrow 0} T_r$ as above for a family of operators $\{T_r\}_{r>0}$. Given $\{t_i\}_i$ a fixed decreasing sequence $t_i \geq t_{i+1} \searrow 0$ we define the **oscillation operator** as

$$\mathcal{O}(Tf)(x) = \left(\sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \varepsilon_{i+1} < \varepsilon_i \leq t_i} |T_{\varepsilon_{i+1}} f(x) - T_{\varepsilon_i} f(x)|^2 \right)^{1/2}.$$

We shall also consider the operator

$$\mathcal{O}'(Tf)(x) = \left(\sum_{i=1}^{\infty} \sup_{t_{i+1} < \delta_i \leq t_i} |T_{t_{i+1}} f(x) - T_{\delta_i} f(x)|^2 \right)^{1/2}.$$

It is easy to see that

$$\mathcal{O}'(Tf) \sim \mathcal{O}(Tf). \tag{0.1}$$

We shall also consider the ρ-**variation operator**

$$\mathcal{V}_\rho(Tf)(x) = \sup_{\varepsilon_i \searrow 0} \left(\sum_{i=1}^{\infty} |T_{\varepsilon_{i+1}} f(x) - T_{\varepsilon_i} f(x)|^\rho \right)^{1/\rho}$$

where the sup is taken over all sequences of real numbers $\{\varepsilon_i\}$ decreasing to zero.

In this paper we study the oscillation and variation of the Riesz transforms $R_j = \partial_{x_j} (-\Delta)^{1/2}$ in $\mathbb{R}^n$. We shall prove the following result.

THEOREM A: *Given p in the range $1 < p < \infty$, $-1 < \alpha < p-1$, and $\rho > 2$, there exist constants $C_\alpha, C_{\alpha,\rho}$ independent of n such that*

$$||\mathcal{O}(R_j f)||_{L^p(\mathbb{R}^n, |x|^\alpha dx)} \leq C_\alpha ||f||_{L^p(\mathbb{R}^n, |x|^\alpha dx)}, \quad j = 1, \ldots, n$$

and

$$||\mathcal{V}_\rho(R_j f)||_{L^p(\mathbb{R}^n, |x|^\alpha dx)} \leq C_{\alpha,\rho} ||f||_{L^p(\mathbb{R}^n, |x|^\alpha dx)}, \quad j = 1, \ldots, n.$$

Consider the family of operators

$$H_r f(x) = \int_{\{|x-y|>r\}} \frac{1}{x-y} f(y) dy$$

in $L^p(\mathbb{R}, dx)$. The following result is proved in [CJRW].

THEOREM 0.2: *Given p in the range $1 < p < \infty$ then*

$$(0.3)\quad \|\mathcal{O}(Hf)\|_{L^p(\mathbb{R})} \leq C_p\|f\|_{L^p(\mathbb{R})}, \text{ and } \|V_\rho(Hf)\|_{L^p(\mathbb{R})} \leq C_p\|f\|_{L^p(\mathbb{R})}, \rho > 2.$$

On the other hand, dimension free results for operators acting on $L^p(\mathbb{R}^n, dx)$ can be found in [S3], [AC], [P], [DR], [R].

We need three steps in order to get Theorem A. The steps, that coincide with the sections of the paper, are as follows.

In section 1, by using (0.3) and a geometrical argument, we prove a pointwise estimate for the sharp maximal functions of the oscillation and variation of the Hilbert transform; see Lemma 1.4. This pointwise estimate, together with some vector-valued Calderón–Zygmund theory, allows us to prove that these particular oscillations and variations are bounded from $L^p(\mathbb{R}, v(x)dx)$ into $L^p(\mathbb{R}, v(x)dx)$ for p in the range $1 < p < \infty$ and v any weight in the Muckenhoupt [Mu] class A_p (see Theorem 1.5). We believe this result is of independent interest. This method of proof follows closely some of the ideas in [RRT]. That is, describe a (possible sublinear) operator, for which a boundedness property is known, through a linear operator valued in a certain Banach space with an associated kernel. The study of the smoothness of the kernel (rather non-trivial in the present case) gives us a pointwise estimate of the sharp maximal function; see inequality (1.3). No information about A_1-boundedness is obtained in this way; see [RRT] for more examples of this type.

In section 2 we prove a weighted transference result for positive operators induced by flows on L^p; see Theorem 2.12 and Remark 2.13. The results are good enough for our purposes, but it is worth mentioning that they can also be proved in the case of strongly continuous one-parameter group of positive invertible linear operators on $L^p(X, \mathcal{F}, \mu)$. The interested reader can fill in the details; see [GT] and [MT] for some related results.

The following comment should be made before describing the content of section 3. Assume that we have a family of operators $\{S_r\}$ that are bounded from $L^p_{B_1}(\mu)$ into $L^p_{B_2}(\mu)$, where $1 < p < \infty$ and B_1, B_2 are Banach spaces, and that the limit $Sf(x) = \lim_{r\searrow 0} S_r f(x)$ exists in some sense. The definition of the oscillation and variation operators can be generalized in the obvious way, that is for example

$$(0.4)\qquad \mathcal{O}(Sf) = \left(\sum_{i=1}^{\infty} \sup_{t_{i+1}\leq\varepsilon_{i+1}<\varepsilon_i\leq t_i} \|S_{\varepsilon_{i+1}}f(x) - S_{\varepsilon_i}f(x)\|^2_{B_2}\right)^{1/2}.$$

An analogous definition can be given for the variation operator.

Denote by $\mathcal{R}$ the $\ell^2(\{1,\dots,n\})$-valued operator given by

$$\mathcal{R}f(x) = (R_1 f(x), \dots, R_n f(x)), \quad R_j = \partial_{x_j}(-\Delta)^{1/2}.$$

It is known that

$$\mathcal{R}f(x) = \lim_{\varepsilon \searrow 0}(R_{\varepsilon,1} f(x), \dots, R_{\varepsilon,n} f(x)),$$

where

$$R_{\varepsilon,j} f(x) = c_n \int_{|x-y|>\varepsilon} \frac{(x_j - y_j)}{|x-y|^{n+1}} f(y)dy,$$

with c_n a specific constant (see (3.1)). Therefore

$$\begin{aligned}
&\mathcal{O}(\mathcal{R}f)(x)\\
&= \Bigg(\sum_{i=1}^{\infty} \sup_{t_{i+1} \le \varepsilon_{i+1} < \varepsilon_i \le t_i} \|\mathcal{R}_{\varepsilon_{i+1}} f(x) - \mathcal{R}_{\varepsilon_i} f(x)\|^2_{\ell^2(\{1,\dots,n\})} \Bigg)^{1/2}\\
&= \Bigg(\sum_{i=1}^{\infty} \sup_{t_{i+1} \le \varepsilon_{i+1} < \varepsilon_i \le t_i} \Bigg[\Bigg\{ \sum_{j=1}^{n} |R_{\varepsilon_{i+1},j} f(x) - \mathcal{R}_{\varepsilon_i,j} f(x)|^2 \Bigg\}^{1/2} \Bigg]^2 \Bigg)^{1/2}\\
&= \Bigg(\sum_{i=1}^{\infty} \sup_{t_{i+1} \le \varepsilon_{i+1} < \varepsilon_i \le t_i} \Bigg[\Bigg\{ \sum_{j=1}^{n} \Big| c_n \int_{\varepsilon_{i+1} < |x-y| < \varepsilon_i} \frac{(x_j - y_j)}{|x-y|^{n+1}} f(y)dy \Big|^2 \Bigg\}^{1/2} \Bigg]^2 \Bigg)^{1/2}.
\end{aligned}$$

Analogous formulas can be obtained for the variation. By using the transference result in section 2 and the weighted norm inequalities in section 1 we prove the following Theorem, which is the main result in section 3 and the motivation of the paper.

THEOREM B: *Given p in the range $1 < p < \infty$,, α with $-1 < \alpha < p - 1$, and $\rho > 2$, there exist constants $C_\alpha, C_{\alpha,\rho}$ independent of n, such that*

$$\|\mathcal{O}(\mathcal{R}f)\|_{L^p(\mathbb{R}^n, |x|^\alpha dx)} \le C_\alpha \|f\|_{L^p(\mathbb{R}^n, |x|^\alpha dx)},$$

and

$$\|\mathcal{V}_\rho(\mathcal{R}f)\|_{L^p(\mathbb{R}^n, |x|^\alpha dx)} \le C_{\alpha,\rho} \|f\|_{L^p(\mathbb{R}^n, |x|^\alpha dx)}.$$

Since $\mathcal{O}(R_j f)(x) \le \mathcal{O}(\mathcal{R}f)(x)$ and $\mathcal{V}_\rho(R_j f)(x) \le \mathcal{V}_\rho(\mathcal{R}f)(x)$ for $j = 1, \dots, n$ and $f \in L^p(\mathbb{R}^n)$, Theorem B contains Theorem A.

1. One-dimensional results

We shall denote by E the mixed norm Banach space of two variable functions h defined on $\mathbb{R} \times \mathbb{N}$ such that

$$(1.1)\qquad \|h\|_E \equiv \left(\sum_i (\sup_s |h(s,i)|)^2 \right)^{1/2} < \infty.$$

Given a fixed decreasing sequence $t_i \geq t_{i+1} \searrow 0$, let $J_i = (t_{i+1}, t_i]$ and define the operator $U(T)$: $f \longrightarrow U(T)f$, where $U(T)f(x)$ is the E-valued function given by

$$U(T)f(x) = \{T_{t_{i+1}} f(x) - T_s f(x)\}_{s \in J_i, i \in \mathbb{N}}.$$

Here the expression $\{T_{t_{i+1}} f(x) - T_s f(x)\}_{s \in J_i, i \in \mathbb{N}}$ is a convenient abbreviation for the element of E given by

$$(s,i) \longrightarrow (T_{t_{i+1}} f(x) - T_s f(x))\chi_{J_i}(s),$$

and $\{T_r\}_{r>0}$ is a family of operators defined on $L^p(\mathbb{R})$ for some p. Then

$$\mathcal{O}'(Tf)(x) = \|\{T_{t_{i+1}} f(x) - T_s f(x)\}_{s \in J_i, i \in \mathbb{N}}\|_E = \|U(T)f(x)\|_E.$$

Therefore we will have inequalities of the type $\|\mathcal{O}'(Tf)\|_{L^p} \leq C\|f\|_{L^p}$ if and only if the operator $U(T)$ defined above maps L^p boundedly into L^p_E.

Let $\Theta = \{\beta : \beta = \{\varepsilon_i\}, \varepsilon_i \in \mathbb{R}, \varepsilon_i \searrow 0\}$. We consider the set $\mathbb{N} \times \Theta$ and denote by F_ρ the mixed norm space of two variables functions $g(i,\beta)$ such that

$$\|g\|_{F_\rho} \equiv \sup_\beta \left(\sum_i |g(i,\beta)|^\rho \right)^{1/\rho} < \infty.$$

We also consider the F_ρ-valued operator $V(T)$: $f \longrightarrow V(T)f$ on $L^p(\mathbb{R})$ given by

$$(1.2)\qquad V(T)f(x) = \{T_{\varepsilon_{i+1}} f(x) - T_{\varepsilon_i} f(x)\}_{\beta \in \Theta, \beta = \{\varepsilon_i\}}.$$

Here the expression $\{T_{\varepsilon_{i+1}} f(x) - T_{\varepsilon_i} f(x)\}_{\beta \in \Theta, \beta = \{\varepsilon_i\}}$ is an abbreviation for the element of F_ρ given by

$$(i,\beta) = (i, \{\varepsilon_k\}) \longrightarrow T_{\varepsilon_{i+1}} f(x) - T_{\varepsilon_i} f(x).$$

As in the case of the oscillation operator it is clear that

$$\mathcal{V}_\rho(Tf) = \|V(T)f\|_{F_\rho},$$

and therefore we will have inequalities of the type $\|\mathcal{V}_\rho(Tf)\|_{L^p} \leq C\|f\|_{L^p}$ if and only if the operator $V(T)$ defined in (1.2) maps L^p boundedly into $L^p_{F_\rho}$.

By the comments above inequality (0.3) is equivalent to saying that for p in the range $1 < p < \infty$ the operators

$$U(H)f(x) = \{H_{t_{i+1}}f(x) - H_s f(x)\}_{s\in J_i, i\in\mathbb{N}} = \left\{ \int_{\{t_{i+1}<|x-y|<s\}} \frac{1}{x-y} f(y)dy \right\}_{s\in J_i, i\in\mathbb{N}}$$

map $L^p(\mathbb{R})$ into $L^p_E(\mathbb{R})$, and the operator

$$V(H)f(x) = \{H_{\varepsilon_{i+1}}f(x) - H_{\varepsilon_i} f(x)\}_{\beta\in\Theta, \beta=\{\varepsilon_i\}} = \left\{ \int_{\{\varepsilon_{i+1}<|x-y|<\varepsilon_i\}} \frac{1}{x-y} f(y)dy \right\}_{\beta\in\Theta, \beta=\{\varepsilon_i\}}$$

maps $L^p(\mathbb{R})$ into $L^p_{F_\rho}(\mathbb{R})$.

Our aim is to prove that $U(H)$ maps $L^p(v)$ into $L^p_E(v)$ and that $V(H)$ maps $L^p(v)$ into $L^p_{F_\rho}(v)$, for p in the range $1 < p < \infty$ and v a weight in the A_p class of Muckenhoupt. It is well known (see [RRT]) that in order to prove such weighted inequalities it is enough to prove that for every $r > 1$ there exists a constant C_r such that

$$(U(H)f)^\sharp(x) \le C_r M_r f(x), \quad (V(H)f)^\sharp(x) \le C_r M_r f(x). \tag{1.3}$$

Recall that, if B is a Banach space and φ is a B-valued function,

$$\varphi^\sharp(x) = \sup_{x\in I} \frac{1}{|I|} \int_I \|\varphi(y) - \frac{1}{|I|}\int_I \varphi(z)dz\|_B dy \quad \text{and}$$
$$M_r\varphi(x) = \left(\sup_{x\in I} \frac{1}{|I|} \int_I \|\varphi(y)\|_B^r dy \right)^{1/r}.$$

It is also well known (see [RRT]) that, in order to prove pointwise inequalities such as (1.3), it is enough to prove the following lemma.

LEMMA 1.4: *For each $r > 1$ and $\rho > 2$ there exist constants $C_r, C_{r,\rho}$ such that, for every interval $I = (x_0 - l, x_0 + l)$, the following inequalities are true:*

$$\frac{1}{|I|} \int_I \|U(H)f(x) - U(H)(f\chi_{4I^c})(x_0)\|_E dx \le C_r M_r f(x_0) \quad \text{and}$$
$$\frac{1}{|I|} \int_I \|V(H)f(x) - V(H)(f\chi_{4I^c})(x_0)\|_{F_\rho} dx \le C_{r,\rho} M_r f(x_0),$$

where $4I^c$ denotes the complementary set of the interval $4I = (x_0 - 4l, x_0 + 4l)$.

Proof: We shall only prove the inequality for the operator $U(H)$; a similar proof can be given for the operator $V(H)$.

Fix f and an interval $I = (x_0 - l, x_o + l)$. Define $f_1(y) = f(y)\chi_{4I}(y)$ and $f_2(y) = f(y) - f_1(y)$. Since the operator $U(H)$ is linear, by using Hölder's inequality and (0.3), we have

$$
\begin{aligned}
&\frac{1}{|I|}\int_I \|U(H)f(x) - U(H)(f\chi_{4I^c})(x_0)\|_E dx \\
=&\frac{1}{|I|}\int_I \|U(H)(f_1)(x) + U(H)(f_2)(x) - U(H)(f_2)(x_0)\|_E dx \\
\leq&\left(\frac{1}{|I|}\int_I \|U(H)(f_1)(x)\|_E^r dx\right)^{1/r} \\
&+\left(\frac{1}{|I|}\int_I \|U(H)(f_2)(x) - U(H)(f_2)(x_0)\|_E dx\right) \\
\leq&C_r\left(\frac{1}{|I|}\int_{\mathbb{R}} |f_1(x)|^r dx\right)^{1/r} + \left(\frac{1}{|I|}\int_I \|U(H)(f_2)(x) - U(H)(f_2)(x_0)\|_E dx\right) \\
=&C_r\left(\frac{1}{|I|}\int_{4I} |f(x)|^r dx\right)^{1/r} + \left(\frac{1}{|I|}\int_I \|U(H)(f_2)(x) - U(H)(f_2)(x_0)\|_E dx\right) \\
\leq&C_r M_r f(x_0) + \left(\frac{1}{|I|}\int_I \|U(H)(f_2)(x) - U(H)(f_2)(x_0)\|_E dx\right).
\end{aligned}
$$

Now we shall dominate the second summand. Let $x \in I$; we have

$$
\begin{aligned}
&\|U(H)(f_2)(x) - U(H)(f_2)(x_0)\|_E \\
=&\Bigg\|\Bigg\{\int_{\{t_{i+1}<|x-y|<s\}} \frac{1}{x-y} f(y)\chi_{4I^c}(y)dy \\
&-\int_{\{t_{i+1}<|x_0-y|<s\}} \frac{1}{x_0-y} f(y)\chi_{4I^c}(y)dy\Bigg\}_{s\in J_i, i\in\mathbb{N}}\Bigg\|_E \\
\leq&\Bigg\|\Bigg\{\int_{\{t_{i+1}<|x-y|<s\}} \Big(\frac{1}{x-y} - \frac{1}{x_0-y}\Big) f(y)\chi_{4I^c}(y)dy\Bigg\}_{s\in J_i, i\in\mathbb{N}}\Bigg\|_E \\
&+\Bigg\|\Bigg\{\int_{\mathbb{R}} (\chi_{\{t_{i+1}<|x-y|<s\}} - \chi_{\{t_{i+1}<|x_0-y|<s\}}) \frac{1}{x_0-y} f(y)\chi_{4I^c}(y)dy\Bigg\}_{s\in J_i, i\in\mathbb{N}}\Bigg\|_E \\
=&A_1 + A_2.
\end{aligned}
$$

Since $\|\{\chi_{\{t_{i+1}<|x-y|<s\}}\}_{s\in J_i, i\in\mathbb{N}}\|_E \leq 1$, by using Minkowsky's inequality we

have

$$\begin{aligned}
A_1 &\le \int_{\mathbb{R}} \|\{\chi_{\{t_{i+1}<|x-y|<s\}}\}_{s\in J_i, i\in\mathbb{N}}\|_E \left|\frac{1}{x-y}-\frac{1}{x_0-y}\right| |f(y)|\chi_{4I^c}(y)dy \\
&\le \int_{\mathbb{R}} \left|\frac{1}{x-y}-\frac{1}{x_0-y}\right| |f(y)|\chi_{4I^c}(y)dy \\
&\le C\int_{\mathbb{R}} \frac{|x-x_0|}{|y-x_0|^2}|f(y)|\chi_{4I^c}(y)dy \\
&\le C\sum_{k=0}^{\infty}\int_{\{2^k4l<|x_0-y|<2^{k+1}4l\}} \frac{|x-x_0|}{|y-x_0|^2}|f(y)|dy \\
&\le C\sum_{k=0}^{\infty}\frac{l}{(2^k4l)^2}\int_{\{|x_0-y|<2^{k+1}4l\}} |f(y)|dy \\
&\le CMf(x_0),
\end{aligned}$$

where Mf is the Hardy–Littlewood maximal function of f.

Now we shall deal with A_2. The integral

$$\int_{\mathbb{R}} |\chi_{\{t_{i+1}<|x-y|<s\}}-\chi_{\{t_{i+1}<|x_0-y|<s\}}|\frac{1}{|x_0-y|}|f(y)|\chi_{4I^c}(y)dy$$

will only be non-zero if either $\chi_{\{t_{i+1}<|x-y|<s\}}=1$ and $\chi_{\{t_{i+1}<|x_0-y|<s\}}=0$ or viceversa. That means the integral will only be non-zero in the following cases:

(i) $t_{i+1}<|x-y|<s$ and $|x_0-y|<t_{i+1}$,

(ii) $t_{i+1}<|x-y|<s$ and $|x_0-y|>s$,

(iii) $t_{i+1}<|x_0-y|<s$ and $|x-y|<t_{i+1}$,

(iv) $t_{i+1}<|x_0-y|<s$ and $|x-y|>s$.

In the first case we observe that, as $|x-x_0|<l$, we have $t_{i+1}<|x-y|\le|x-x_0|+|x_0-y|<l+t_{i+1}$. Analogously, in the third case we have $t_{i+1}<|x_0-y|<l+t_{i+1}$. In the second case we have $s<|x_0-y|\le|x_0-x|+|x-y|<l+s$ and analogously in the fourth we have $s<|x-y|<l+s$. Therefore we have

$$\begin{aligned}
&\int_{\mathbb{R}} |\chi_{\{t_{i+1}<|x-y|<s\}}-\chi_{\{t_{i+1}<|x_0-y|<s\}}|\frac{1}{|x_0-y|}|f(y)|\chi_{4I^c}(y)dy \\
&\le \int_{\mathbb{R}} \chi_{\{t_{i+1}<|x-y|<t_{i+1}+l\}}\chi_{\{t_{i+1}<|x-y|<s\}}\frac{1}{|x_0-y|}|f(y)|\chi_{4I^c}(y)dy \\
&\quad+\int_{\mathbb{R}} \chi_{\{s<|x_0-y|<s+l\}}\chi_{\{t_{i+1}<|x-y|<s\}}\frac{1}{|x_0-y|}|f(y)|\chi_{4I^c}(y)dy \\
&\quad+\int_{\mathbb{R}} \chi_{\{t_{i+1}<|x_0-y|<t_{i+1}+l\}}\chi_{\{t_{i+1}<|x_0-y|<s\}}\frac{1}{|x_0-y|}|f(y)|\chi_{4I^c}(y)dy \\
&\quad+\int_{\mathbb{R}} \chi_{\{s<|x-y|<s+l\}}\chi_{\{t_{i+1}<|x_0-y|<s\}}\frac{1}{|x_0-y|}|f(y)|\chi_{4I^c}(y)dy
\end{aligned}$$

$$\begin{aligned}&\leq C\bigg(\int_{\mathbb{R}}\chi_{\{t_{i+1}<|x-y|<s\}}\frac{1}{|x_0-y|^r}|f(y)|^r\chi_{4I^c}(y)dy\bigg)^{1/r}l^{1/r'}\\&\quad+C\bigg(\int_{\mathbb{R}}\chi_{\{t_{i+1}<|x_0-y|<s\}}\frac{1}{|x_0-y|^r}|f(y)|^r\chi_{4I^c}(y)dy\bigg)^{1/r}l^{1/r'},\end{aligned}$$

where in the last inequality we have used Hölder's inequality with r in the range $1<r<\infty$. Returning to our estimation of A_2, we have

$$\begin{aligned}A_2=&\bigg\|\bigg\{\int_{\mathbb{R}}|\chi_{\{t_{i+1}<|x-y|<s\}}-\chi_{\{t_{i+1}<|x_0-y|<s\}}|\\&\times\frac{1}{|x_0-y|}|f(y)|\chi_{4I^c}(y)dy\bigg\}_{s\in J_i,i\in\mathbb{N}}\bigg\|_E\\\leq&Cl^{1/r'}\bigg\|\bigg\{\bigg(\int_{\mathbb{R}}\chi_{\{t_{i+1}<|x-y|<s\}}\frac{1}{|x_0-y|^r}|f(y)|^r\chi_{4I^c}(y)dy\bigg)^{1/r}\bigg\}_{s\in J_i,i\in\mathbb{N}}\bigg\|_E\\&+Cl^{1/r'}\|\{(\int_{\mathbb{R}}\chi_{\{t_{i+1}<|x_0-y|<s\}}\frac{1}{|x_0-y|^r}|f(y)|^r\chi_{4I^c}(y)dy)^{1/r}\}_{s\in J_i,i\in\mathbb{N}}\|_E\\=&A_{21}+A_{22}.\end{aligned}$$

Choosing $1<r<2$ we have

$$\begin{aligned}&\bigg\|\bigg\{\bigg(\int_{\mathbb{R}}\chi_{\{t_{i+1}<|x-y|<s\}}\frac{1}{|x_0-y|^r}|f(y)|^r\chi_{4I^c}(y)dy\bigg)^{1/r}\bigg\}_{s\in J_i,i\in\mathbb{N}}\bigg\|_E\\=&\bigg(\sum_{i\in\mathbb{N}}\sup_{s\in J_i}\bigg(\int_{\mathbb{R}}\chi_{\{t_{i+1}<|x-y|<s\}}\frac{1}{|x_0-y|^r}|f(y)|^r\chi_{4I^c}(y)dy\bigg)^{2/r}\bigg)^{1/2}\\\leq&\bigg(\sum_{i\in\mathbb{N}}\bigg(\int_{\mathbb{R}}\chi_{\{t_{i+1}<|x-y|<t_i\}}\frac{1}{|x_0-y|^r}|f(y)|^r\chi_{4I^c}(y)dy\bigg)^{2/r}\bigg)^{1/2}\\\leq&\bigg(\sum_{i\in\mathbb{N}}\int_{\mathbb{R}}\chi_{\{t_{i+1}<|x-y|<t_i\}}\frac{1}{|x_0-y|^r}|f(y)|^r\chi_{4I^c}(y)dy\bigg)^{1/r}\\=&\bigg(\int_{\mathbb{R}}\sum_{i\in\mathbb{N}}\chi_{\{t_{i+1}<|x-y|<t_i\}}\frac{1}{|x_0-y|^r}|f(y)|^r\chi_{4I^c}(y)dy\bigg)^{1/r}\\\leq&\bigg(\int_{\mathbb{R}}\frac{1}{|x_0-y|^r}|f(y)|^r\chi_{4I^c}(y)dy\bigg)^{1/r}\\\leq&\bigg(\sum_{k=0}^{\infty}\int_{\{2^k4l<|x_0-y|<2^{k+1}4l\}}\frac{1}{|x_0-y|^r}|f(y)|^rdy\bigg)^{1/r}\\\leq&\bigg(\sum_{k=0}^{\infty}\frac{1}{(2^k4l)^r}\int_{\{|x_0-y|<2^{k+1}4l\}}|f(y)|^rdy\bigg)^{1/r}\end{aligned}$$

$$\leq C_r\left(\sum_{k=0}^{\infty}\frac{1}{(2^k4l)^{r-1}}M(f^r)(x_0)\right)^{1/r}$$
$$\leq C_r(M(f^r)(x_0))^{1/r}l^{-1/r'}.$$

Therefore we get $A_{21} \leq C_r(M(f^r)(x_0))^{1/r}$. Analogously we get

$$A_{22} \leq C_r(M(f^r)(x_0))^{1/r}$$

and this is the end of the proof of the Lemma. ∎

As remarked earlier, the preceding lemma gives the inequality (1.3) and therefore as a consequence we have the following Theorem.

THEOREM 1.5: *The operators $\mathcal{O}(H)$ and $\mathcal{V}_\rho(H), \rho > 2$ map $L^p(\mathbb{R}, v)$ boundedly into itself for $1 < p < \infty$ and v a weight in the A_p class of Muckenhoupt.*

2. Weighted transference

Given a σ-finite measure space $(X, \mathcal{F}, \mu)$, an endomorphism of the σ-algebra $\mathcal{F}$ modulo null sets is a set function $\Phi: \mathcal{F} \to \mathcal{F}$ which satisfies

(i) $\Phi(\bigcup_n E_n) = \bigcup_n \Phi(E_n)$, for disjoint $E_n \in \mathcal{F}, n = 1, 2, \ldots$;
(ii) $\Phi(X \setminus E) = \Phi(X) \setminus \Phi(E)$, for all $E \in \mathcal{F}$;
(iii) given $E \in \mathcal{F}$, with $\mu(E) = 0$, then $\mu(\Phi E) = 0$.

In these circumstances, Φ induces a unique positive and multiplicative linear operator, also denoted by Φ, on the space of (finite-valued or extended) measurable functions such that

$$\Phi(f_n) \to \Phi(f)\mu\text{-a.e. whenever } 0 \leq f_n \to f, \mu\text{-a.e.} \tag{2.1}$$

The action of Φ on simple functions is given by

$$\Phi(\sum_i c_i\chi_{E_i})(x) = \sum_i c_i\chi_{\Phi(E_i)}(x), \quad c_i \in \mathbb{C}.$$

Given a Banach space B, Φ has an extension to the simple B-valued functions, also denoted by Φ, given by

$$\Phi(\sum_i \chi_{E_i}b_i) = \sum_i \chi_{\Phi(E_i)}b_i \quad (b_i \in B, E_i \in \mathcal{F}).$$

It is clear that, for $f: X \to B$ a simple function,

$$\|\Phi(f)(x)\|_B = \Phi(\|f\|_B)(x) \quad (x \in X). \tag{2.2}$$

In other words, if Φ induces an operator T bounded in $L^p(\mu)$, then T has a bounded extension, also denoted by T, from $L^p_B(\mu)$ into $L^p_B(\mu)$ for any Banach space B. The action of T on $L^p(\mu) \otimes B$ is defined as

$$T\Big(\sum_i \varphi_i b_i\Big) = \sum_i T(\varphi_i) b_i, \quad b_i \in B, \varphi_i \in L^p(\mu). \tag{2.3}$$

The norm of T on $L^p_B(\mu)$ equals the norm of T on $L^p(\mu)$.

STANDING HYPOTHESES 2.4: *Throughout, we take $(X, \mathcal{F}, \mu)$ to be a σ-finite measure space and $\mathcal{T} = \{T^t : t \in \mathbb{R}\}$ a strongly continuous one-parameter group of positive invertible linear operators on $L^p = L^p(X, \mathcal{F}, \mu)$, for some fixed p in the range $1 < p < \infty$, such that for each $t \in \mathbb{R}$, there exists a σ-endomorphism, Φ^t, with $T^t f = \Phi^t f$. In this case we shall say that $\mathcal{T}$ satisfies $\mathbf{SH}_p$.*

From the group structure of $\mathcal{T}$, it follows that for each $t \in \mathbb{R}$, there exists a positive function J_t such that

$$J_{t+s} = J_t \Phi^t J_s \quad \text{and} \quad \int_X J_t \Phi^t f \, d\mu = \int_X f \, d\mu, \quad t, s \in \mathbb{R}. \tag{2.5}$$

Using the properties of Bochner integration we have

$$T^t\Big(\int_K T^s f \, ds\Big) = \int_K T^t(T^s f) \, ds, \quad t \in \mathbb{R} \tag{2.6}$$

for all $f \in L^p(\mu)$ and all compact subsets K of $\mathbb{R}$.

Definition 2.7: Let $(X, \mathcal{F}, \mu)$, $\mathcal{T}$ and fixed p in the range $1 < p < \infty$ be as in the $\mathbf{SH}_p$ 2.4, and let ω be a measurable function on X such that $\omega(x) > 0$, μ-almost everywhere. We shall say that ω is an **Ergodic A_p-weight** with respect to the group $\mathcal{T}$ if, for μ-almost all $x \in X$, the function $t \to J_t(x)\Phi^t(\omega)(x)$ is an A_p weight with an A_p-constant independent of x, where J_t and Φ^t are as in (2.5).

We shall denote by $E_p(\mathcal{T})$ the class of ergodic A_p-weights associated with the group $\mathcal{T}$. Given a weight ω and a family $\mathcal{T}$ satisfying $\mathbf{SH}_p$ 2.4, we shall use the following notation:

$$\mathcal{T}\omega_x(t) = J_t(x)\Phi^t(\omega)(x). \tag{2.8}$$

Definition 2.9: Any x-independent A_p-constant for the family of weights $\{\mathcal{T}\omega_x\}$ will be referred to as an $E_p(\mathcal{T})$-constant for ω.

In [GT] we develop a weighted ergodic theory; one of the results obtained there is the following.

THEOREM 2.10: *Let $\mathcal{T}$ be a family of operators satisfying* $\mathbf{SH}_p$ 2.4 *for every p in the range $1 < p < \infty$. Assume that K is a sublinear operator such that $\|Kf\|_{L^{p_0}(\omega d\mu)} \leq C_\omega \|f\|_{L^{p_0}(\omega d\mu)}$ for every $\omega \in E_{p_0}(\mathcal{T})$, where p_0 is fixed in the range $1 < p < \infty$ and the constant C_ω only depends on an $E_{p_0}(\mathcal{T})$-constant for ω. Then K is bounded from $L^p(\omega d\mu)$ into $L^p(\omega d\mu)$ for every $p, 1 < p < \infty$, and every $\omega \in E_p(\mathcal{T})$.*

Definition 2.11: Given Banach spaces B_1, B_2, and a function

$$k \in L^1_{loc,\mathcal{L}(B_1,B_2)}(\mathbb{R}),$$

we shall say that k is a "bounded oscillation kernel" if there exists an operator K with the following properties.

(i) K maps $L^p_{B_1}(\mathbb{R}, v)$ into $L^p_{B_2}(\mathbb{R}, v)$, for every $v \in A_p$.

(ii) If $\varphi \in L^\infty_{B_1}(\mathbb{R})$ and has compact support, then

$$K\varphi(t) = \int_{\mathbb{R}} k(t-s)\varphi(s)ds, \quad t \notin \text{support of } \varphi.$$

(iii) Given $\{t_i\}_i$ any deceasing sequence $t_i \geq t_{i+1} \searrow 0$, the oscillation operator

$$\mathcal{O}(K\varphi)(x) = \left(\sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \varepsilon_{i+1} < \varepsilon_i \leq t_i} \|K_{\varepsilon_i, \varepsilon_{i+1}} \varphi(x)\|^2_{B_2} \right)^{1/2}$$

is bounded from $L^p_{B_1}(\mathbb{R}, v)$ into $L^p(\mathbb{R}, v)$, for all p in the range $1 < p < \infty$ and each $v \in A_p$, where

$$K_{\varepsilon,\varepsilon'}\varphi(t) = \int_{\{s:\varepsilon<|t-s|<\varepsilon'\}} k(t-s)\varphi(s)ds, \quad 0 < \varepsilon < \varepsilon'.$$

Moreover, the operator norm depends only on any A_p constant of v.

Now we state the main theorem in this section. Recall that T^t has a natural extension to $L^p_{B_1}(X, d\mu)$, also denoted by T^t (see (2.3)).

THEOREM 2.12: *Let p be in the range $1 < p < \infty$ and let $\mathcal{T}$ be a family of operators satisfying* $\mathbf{SH}_p$ 2.4. *Let B_1, B_2 be Banach spaces and k a "bounded oscillation kernel" as in Definition 2.11. Given a decreasing sequence $t_i \geq t_{i+1} \searrow 0$, we consider finite sets $L \subset \mathbb{N}$ and $\mathcal{J}_i \subset (t_{i+1}, t_i]$. We define the operator $\mathcal{O}_K^{L,\mathcal{J}_i}$ on $L^p_{B_1}(X, \mu)$ by*

$$\mathcal{O}_K^{L,\mathcal{J}_i} f(x) = \left(\sum_{i\in L} \max_{\varepsilon_i, \varepsilon_{i+1} \in \mathcal{J}_i} \left\| \int_{\{\varepsilon_{i+1}<|s|<\varepsilon_i\}} k(s) T^{-s} f(x) ds \right\|^2_{B_2} \right)^{1/2}.$$

Then

$$\sup_{\{L,\mathcal{J}_i: L,\mathcal{J}_i finite\}} \|O_K^{L,\mathcal{J}_i} f\|_{L^p(X,\omega)} \le N_p(\mathcal{O}(K),\mathcal{T}\omega)\|f\|_{L^p_{B_1}(X,\omega)},$$

for every $\omega \in E_p(\mathcal{T})$. Here $N_p(\mathcal{O}(K),\mathcal{T}\omega)$ denotes an essential bound relative to x of the operator-norm of $\mathcal{O}(K)$ as a bounded operator from $L^p_{B_1}(\mathbb{R},\mathcal{T}\omega_x)$ into $L^p(\mathbb{R},\mathcal{T}\omega_x)$, where $\mathcal{T}\omega_x(t)$ is defined in (2.8).

We observe that as $\omega \in E_p(\mathcal{T})$ then $\mathcal{T}\omega_x(.) \in A_p$ with an A_p constant independent of x, so that such essential bounds exist.

Proof: Given $f \in L^p_{B_1}(X,\mu)$, define

$$K_{\varepsilon,\varepsilon'} f = \int_{\{\varepsilon<|s|<\varepsilon'\}} k(s)T^{-s} f ds.$$

Observe that since k is integrable on $\{\varepsilon < |t| < \varepsilon'\}$ and $t \to T^t f$ is strongly continuous, the operator $K_{\varepsilon,\varepsilon'}$ is well defined for $f \in L^p(d\mu)$ via Bochner integration. Then, given a decreasing sequence $t_i \ge t_{i+1} \searrow 0$, and finite subsets $L \subset \mathbb{N}$ and $\mathcal{J}_i \subset (t_{i+1}, t_i]$ we have to show that

$$\int_X \|\{K_{\varepsilon_{i+1},\varepsilon_i} f(x)\}_{i\in L,\varepsilon_{i+1}<\varepsilon_i\in\mathcal{J}_i}\|_E^p \omega(x)d\mu(x) \le N_p(\mathcal{O}(K),\mathcal{T}\omega)\int_X \|f(x)\|_{B_1}^p \omega(x)d\mu(x),$$

where E is the Banach space defined in (1.1). By using identities (2.5) and (2.2) we have

$$\begin{aligned}\int_X \|\{K_{\varepsilon_{i+1},\varepsilon_i} f(x)\}\|_E^p \omega(x)d\mu(x) &= \int_X J_t(x)\Phi^t(\|\{K_{\varepsilon_{i+1},\varepsilon_i} f(x)\}\|_E)^p(x)\Phi^t(\omega)(x)d\mu(x)\\ &= \int_X J_t(x)\|\{\Phi^t(K_{\varepsilon_{i+1},\varepsilon_i} f)(x)\}\|_E^p \Phi^t(\omega)(x)d\mu(x).\end{aligned}$$

Let $H \subset \mathbb{R}$ be a compact set such that

$$H \supset \{t : \varepsilon_{i+1} < |t| < \varepsilon_i, \text{ for some } i \in L \text{ and some } \varepsilon_i, \varepsilon_{i+1} \in J_i\}.$$

Let $\delta > 0$ and choose a relatively compact open set $V \subset \mathbb{R}$ such that

$$|V - H|/|V| < 1 + \delta,$$

where $|H|$ is Lebesgue measure of H and $V - H = \{t - s : t \in V, s \in H\}$. Averaging over V, using Fubini's Theorem, the properties of the family $\mathcal{T}$, the fact that $\mathcal{O}(K)$ is bounded in $L^p(\mathbb{R}, v)$ for $v \in A_p$ and the definition of $E_p(\mathcal{T})$ weights, we have

$$
\begin{aligned}
&\int_X \|\{K_{\varepsilon_{i+1},\varepsilon_i} f(x)\}\|_E^p \omega(x) d\mu(x)\\
=&\frac{1}{|V|}\int_V\int_X J_t(x)\|\{\Phi^t(K_{\varepsilon_{i+1},\varepsilon_i} f)(x)\}\|_E^p \Phi^t(\omega)(x)d\mu(x)dt\\
=&\int_X \frac{1}{|V|}\int_V J_t(x)\|\{\Phi^t(K_{\varepsilon_{i+1},\varepsilon_i} f)(x)\}\|_E^p \Phi^t(\omega)(x)dtd\mu(x)\\
=&\int_X \frac{1}{|V|}\int_V \|\{T^t(K_{\varepsilon_{i+1},\varepsilon_i} f)(x)\}\|_E^p J_t(x)\Phi^t(\omega)(x)dtd\mu(x)\\
=&\int_X \frac{1}{|V|}\int_V \|\{\int_{\{s:\varepsilon_{i+1}<|s|<\varepsilon_i\}} k(s)T^{t-s}f(x)ds\}\|_E^p J_t(x)\Phi^t(\omega)(x)dtd\mu(x)\\
=&\int_X \frac{1}{|V|}\int_V \|\{\int_{\{s:\varepsilon_{i+1}<|s|<\varepsilon_i\}} \chi_{V-H}(t-s)k(s)T^{t-s}f(x)ds\}\|_E^p\\
&\times J_t(x)\Phi^t(\omega)(x)dtd\mu(x)\\
\le&\int_X \frac{1}{|V|}\int_{\mathbb{R}} \mathcal{O}(k(\chi_{V-H}(\cdot)T^{(\cdot)}f(x))(t)^p J_t(x)\Phi^t(\omega)(x)dtd\mu(x)\\
\le&\int_X N_p(\mathcal{O}(K),\mathcal{T}\omega)^p\frac{1}{|V|}\int_{\mathbb{R}} \|\chi_{V-H}(t)T^t f(x)\|_{B_1}^p J_t(x)\Phi^t(\omega)(x)dtd\mu(x)\\
=&N_p(\mathcal{O}(K),\mathcal{T}\omega)^p\frac{1}{|V|}\int_{\mathbb{R}}(\chi_{V-H}(t)\int_X \Phi^t(\|f\|_{B_1}^p\omega)(x)J_t(x)d\mu(x)dt\\
=&N_p(\mathcal{O}(K),\mathcal{T}\omega)^p\frac{|V-H|}{|V|}\int_X \|f(x)\|_{B_1}^p\omega(x)d\mu(x)\\
\le&N_p(\mathcal{O}(K),\mathcal{T}\omega)^p(1+\delta)\int_X \|f(x)\|_{B_1}^p\omega(x)d\mu(x).
\end{aligned}
$$

Now let $\delta \to 0$ to obtain the required result. ∎

Remark 2.13: One can define analogously the concept of "bounded ρ-variation kernel" by changing (iii) in Definition 2.11 to

(iii)′ The ρ-variation operator

$$\mathcal{V}_\rho(K\varphi)(x) = \sup_{\{\varepsilon_j\}}\left(\sum_{i=1}^{\infty}\|K_{\varepsilon_i,\varepsilon_{i+1}}\varphi(x)\|_{B_2}^\rho\right)^{1/\rho},$$

where the supremum is taken over all decreasing sequences $\{\varepsilon_i\}$ with $\lim \varepsilon_i = 0$, is bounded from $L^p_{B_1}(\mathbb{R}, v)$ into $L^p(\mathbb{R}, v)$, for all p in the range

$1 < p < \infty$ and each $v \in A_P$. Moreover, the operator norm depends only on any A_p constant of v.

Then the following result, analogous to Theorem 2.12, can be proved.

Let $1 < p < \infty$ and let $\mathcal{T}$ be a family of operators satisfying $\mathbf{SH}_p$ 2.4. Let B_1, B_2, be Banach spaces and k a "bounded ρ-variation kernel", with $\rho > 2$. Let $\mathcal{L}$ be a finite set of decreasing sequences $\{\varepsilon_i\}$ with $\lim \varepsilon_i = 0$. We define the operator $\mathcal{V}^{\mathcal{L}}_{\rho,K}$ on $L^p_{B_1}(X,\mu)$ by

$$\mathcal{V}^{\mathcal{L}}_K f(x) = \max_{\{\varepsilon_i\}\in\mathcal{L}} \left(\left\| \int_{\{\varepsilon_{i+1}<|s|<\varepsilon_i\}} k(s) T^{-s} f(x) ds \right\|^{\rho}_{B_2} \right)^{1/\rho}.$$

Then

$$\sup_{\{\mathcal{L}:\mathcal{L} finite\}} \|\mathcal{V}^{\mathcal{L}}_{\rho,K} f\|_{L^p(X,\omega)} \le N_p(\mathcal{V}_\rho(K), \mathcal{T}\omega) \|f\|_{L^p_{B_1}(X,\omega)},$$

for every $\omega \in E_p(\mathcal{T})$. Here $N_p(\mathcal{V}_\rho(K), \mathcal{T}\omega)$ denotes an essential bound relative to x of the operator-norm of $\mathcal{V}_\rho(K)$ as a bounded operator from $L^p_{B_1}(\mathbb{R}, \mathcal{T}\omega_x)$ into $L^p(\mathbb{R}, \mathcal{T}\omega_x)$, where $\mathcal{T}\omega_x(t)$ is defined in (2.8).

3. Applications to dimension free estimates

The operators $R_j = \partial_{x_j}(-\Delta)^{-\frac{1}{2}}$ are defined, for functions whose Fourier transforms have compact support, by the formula

$$(\partial_{x_j}(-\Delta)^{-\frac{1}{2}} f)\hat{}(\xi) = 2\pi i \xi_j |\xi|^{-1} \hat{f}(\xi);$$

see [S1]. Since Δ is the infinitesimal generator of the Gauss semigroup, the operator $(-\Delta)^{-\frac{1}{2}}$ can also be defined, in terms of the semigroup, as

$$(-\Delta)^{-\frac{1}{2}} = \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty e^{t\Delta} t^{\frac{1}{2}} \frac{dt}{t};$$

see [S2]. Therefore, by using the duality in $L^2(\mathbb{R}^n)$, the kernels associated in the sense of 2.11 with the operators $R_j = \partial_{x_j}(-\Delta)^{-\frac{1}{2}}$, as defined above, can be computed. In fact, if f is a smooth compactly supported function, for all x outside the support of f we have

$$\begin{aligned} (3.1) \quad R_j f(x) &= \partial_{x_j} \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\Big(-\frac{|x-y|^2}{4t}\Big) f(y) dy t^{\frac{1}{2}} \frac{dt}{t} \\ &= -\frac{2\Gamma(\frac{n+1}{2})}{\omega_{n-1}\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})} \int_{\mathbb{R}^n} \frac{(x_j-y_j)}{|x-y|^{n+1}} f(y) dy, \end{aligned}$$

where $\omega_{n-1} = 2\pi^{n/2}/\Gamma(\frac{n}{2})$ is the surface area of the unit sphere in $\mathbb{R}^n$.

Before coming to the proof of Theorem B, we need some preliminary work to set the stage. Then we apply the ergodic A_p theory developed in section 2. We shall use some ideas in [DR], [AC] and [P].

Let k be a "bounded oscillation kernel" with corresponding operator K (see Definition 2.11). We consider the unit sphere Σ_{n-1} of $\mathbb{R}^n$ endowed with the rotationally invariant measure $d\sigma$ normalized so that $\int_{\Sigma_{n-1}} d\sigma = 1$. Given a fixed $y' \in \Sigma_{n-1}$ we consider the one parameter group of operators $\mathcal{T}_{y'} = \{\Phi^t_{y'}\}_t$, where

$$\Phi^t_{y'}(f)(x) = f(x + ty'), \quad x \in \mathbb{R}^n, t \in \mathbb{R}.$$

Clearly $\|\Phi^t_{y'}(f)\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)}$. Therefore, if we define

$$K_{\varepsilon,\varepsilon',y'} = \int_{\{\varepsilon<|s|<\varepsilon'\}} k(s)\Phi^{-s}_{y'} ds, \tag{3.2}$$

then, by Theorem 2.12, we have

$$\|\{K_{\varepsilon_{i+1},\varepsilon_i,y'}f\}_{i\in L;\varepsilon_{i+1},\varepsilon_i\in\mathcal{J}_i}\|_{L^p_E(\mathbb{R}^n,\omega)} \le N_p(\mathcal{O}(K),\mathcal{T}_{y'}\omega)\|f\|_{L^p(\mathbb{R}^n,\omega)} \tag{3.3}$$

for all finite sets $L \subset \mathbb{N}$ and $\mathcal{J}_i \subset (t_{i+1}, t_i]$, and every $\omega \in E_p(\mathcal{T}_{y'})$, where $1 < p < \infty$.

Let P be the projection of the space $L^2(d\sigma)$ into the subspace $\mathcal{H}$ of $L^2(d\sigma)$ generated by the functions $y'_1, \dots, y'_n$.

LEMMA 3.4: *With the notations in (3.2), we have*

$$P(K_{\varepsilon,\varepsilon',\cdot}f(x))(y') = \sum_{j=1}^{n} K^j_{\varepsilon,\varepsilon'}f(x)Y_j(y'), \quad f \in L^\infty,$$

where

$$K^j_{\varepsilon,\varepsilon'}f(x) = \frac{1}{\omega_{n-1}} \int_{\{z\in\mathbb{R}^n:\varepsilon<|z|<\varepsilon'\}} \frac{k(|z|) - k(-|z|)}{|z|^{n-1}} f(x-z) Y_j\Big(\frac{z}{|z|}\Big) dz, \quad j = 1, \dots, n$$

and $\{Y_j\}_{j=1}^n$ are the functions $Y_j(y') = n^{1/2}y'_j$ for $y' \in \Sigma_{n-1}$.

Proof: As P is a projection and $Y_1, \dots, Y_n$ are orthonormal in $L^2(\Sigma_{n-1}, d\sigma)$, we have

$$P(K_{\varepsilon,\varepsilon',\cdot}f(x))(y') = \sum c_j(x)Y_j(y'),$$

where

$$c_j(x) = \int_{\Sigma_{n-1}} K_{\varepsilon,\varepsilon',y'} f(x) Y_j(y') d\sigma(y').$$

By using polar coordinates and the fact that the Y_j's are odd functions, we have that for $f \in L^\infty$,

$$\begin{aligned}
c_j(x) &= \int_{\Sigma_{n-1}} \int_{\{t\in\mathbb{R}:\varepsilon<|t|<\varepsilon'\}} k(t)\Phi_{y'}^{-t} f(x) dt Y_j(y') d\sigma(y') \\
&= \int_{\{t\in\mathbb{R}:\varepsilon<|t|<\varepsilon'\}} \int_{\Sigma_{n-1}} k(t) f(x-ty') Y_j(y') d\sigma(y') dt \\
&= \int_0^\infty \int_{\Sigma_{n-1}} k(t)\chi_{\{t\in\mathbb{R}:\varepsilon<|t|<\varepsilon'\}} f(x-ty') Y_j(y') d\sigma(y') dt \\
&\quad - \int_{-\infty}^0 \int_{\Sigma_{n-1}} k(t)\chi_{\{t\in\mathbb{R}:\varepsilon<|t|<\varepsilon'\}} f(x+ty') Y_j(y') d\sigma(y') dt \\
&= \int_0^\infty (k(t)-k(-t))\chi_{\{t\in\mathbb{R}:\varepsilon<t<\varepsilon'\}} \int_{\Sigma_{n-1}} f(x-ty') Y_j(y') d\sigma(y') dt \\
&= \int_0^\infty (k(t)-k(-t))\chi_{\{t\in\mathbb{R}:\varepsilon<t<\varepsilon'\}} \int_{\Sigma_{n-1}} \frac{f(x-ty')}{t^{n-1}} Y_j(y') t^{n-1} d\sigma(y') dt \\
&= \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{(k(|z|)-k(-|z|))}{|z|^{n-1}} \chi\{z\in\mathbb{R}^n : \varepsilon<|z|<\varepsilon'\} f(x-z) Y_j\Big(\frac{z}{|z|}\Big) dz.
\end{aligned}$$

∎

THEOREM 3.5: *Let k be a "bounded oscillation kernel" on $\mathbb{R}$ as in 2.11. Let $1<p<\infty$; assume that ω is a weight in $\mathbb{R}^n$ such that the function $t \to \Phi_{y'}^t \omega(x)$ is a weight in $A_p(\mathbb{R})$ with an A_p-constant independent of y' and x. Then there exists a constant $C(\omega)$ such that*

$$\Big\| \Big\{ \Big(\sum_{j=1}^n |K^j_{\varepsilon_{i+1},\varepsilon_i} f|^2 \Big)^{1/2} \Big\}_{i\in L;\varepsilon_{i+1},\varepsilon_i\in\mathcal{J}_i} \Big\|_{L^p_E(\mathbb{R}^n,\omega)} \le C(\omega)\|f\|_{L^p(\mathbb{R}^n,\omega)} \tag{3.6}$$

for all finite sets $L \subset \mathbb{N}$ and $\mathcal{J}_i \subset (t_{i+1}, t_i]$. Moreover, the constant C can be taken to be an upper bound for the norm of operators of the form $\mathcal{O}(K)$: $L^p(\mathbb{R}, v) \to L^p(\mathbb{R}, v)$, where $v(t) = \Phi_{y'}^t \omega(x)$ for some y' and x.

Proof: We observe that by using Theorem 2.10 it is enough to prove inequality (3.6) for some $p, 1<p<\infty$. We shall prove it for $p=2$. In fact, by using orthogonality and the representation formula for P in Lemma 3.4 we have

$$
\begin{aligned}
&\left\|\left\{\left(\sum_{j=1}^{n}|K^j_{\varepsilon_{i+1},\varepsilon_i}f|^2\right)^{1/2}\right\}_{i\in L;\varepsilon_{i+1},\varepsilon_i\in\mathcal{J}_i}\right\|_{L^2_E(\mathbb{R}^n,\omega)}\\
&=\left\|\left\{\left(\int_{\Sigma_{n-1}}|\sum_{j=1}^{n}K^j_{\varepsilon_{i+1},\varepsilon_i}fY_j(y')|^2\right)d\sigma(y'))^{1/2}\right\}_{i\in L;\varepsilon_{i+1},\varepsilon_i\in\mathcal{J}_i}\right\|_{L^2_E(\mathbb{R}^n,\omega)}\\
&=\left\|\left\{\left(\int_{\Sigma_{n-1}}|P(K_{\varepsilon_{i+1},\varepsilon_i,\cdot}f(\cdot))(y)|^2d\sigma(y')\right)^{1/2}\right\}_{i\in L;\varepsilon_{i+1},\varepsilon_i\in\mathcal{J}_i}\right\|_{L^2_E(\mathbb{R}^n,\omega)}\\
&\le\left\|\left\{\left(\int_{\Sigma_{n-1}}|K_{\varepsilon_{i+1},\varepsilon_i,y'}f(\cdot)\right)|^2d\sigma(y'))^{1/2}\right\}_{i\in L;\varepsilon_{i+1},\varepsilon_i\in\mathcal{J}_i}\right\|_{L^2_E(\mathbb{R}^n,\omega)}\\
&\le\left(\int_{\Sigma_{n-1}}(\|\{K_{\varepsilon_{i+1},\varepsilon_i,y'}f(\cdot)\}_{i\in L;\varepsilon_{i+1},\varepsilon_i\in\mathcal{J}_i}\|_{L^2_E(\mathbb{R}^n,\omega)})^2d\sigma(y')\right)^{1/2}\\
&\le\left(\int_{\Sigma_{n-1}}N_2(\mathcal{O}(K),\mathcal{T}_{y'}\omega)^2\|f\|^2_{L^2(\mathbb{R}^n,\omega)}d\sigma(y')\right)^{1/2}\\
&\le C(\omega)\|f\|_{L^2(\mathbb{R}^n,\omega))},
\end{aligned}
$$

where in the penultimate inequality we have used (3.3). ∎

Corollary 3.7: *Let $1<p<\infty$ and let $-1<\alpha<p-1$. Let k be a bounded oscillation kernel on $\mathbb{R}$ as in 2.11 and consider the operators $K^j_{\varepsilon,\varepsilon'}$ defined in Lemma 3.4. Then there exists a constant $C_{\alpha,p}$ such that*

$$
\begin{aligned}
\int_{\mathbb{R}^n}\left(\sum_{i=1}^{\infty}\sup_{t_{i+1}<\varepsilon_{i+1}<\varepsilon_i\le t_i}\left(\sum_{j=1}^{n}|K^j_{\varepsilon_{i+1},\varepsilon_i}f(x)|^2\right)\right)^{p/2}|x|^\alpha dx&\\
\le C_{\alpha,p}\int_{\mathbb{R}^n}|f(x)|^p|x|^\alpha dx,&
\end{aligned}
\tag{3.8}
$$

for each sequence $\{t_i\}_i$ such that $t_i>t_{i+1}\searrow 0$.

Proof: In order to apply Theorem 3.5, it will be enough to show that, given $x\in\mathbb{R}^n$ and $y'\in\Sigma_{n-1}$, the function $t\to|x+ty'|^\alpha$ is an A_p-weight on $\mathbb{R}$, with an A_p-constant independent of x and y'. Fix $x\in\mathbb{R}^n, y'\in\Sigma_{n-1}$ and decompose x as $x=x_1+t_0y'$, with $x_1\perp y'$. Then, as $|y'|=1$, we have $|x+ty'|=(|x_1|^2+|t_0+t|^2)^{1/2}\sim|x_1|+|t_0+t|$. Therefore $|x+ty'|^\alpha\sim|x_1|^\alpha+|t_0+t|^\alpha$. Hence if M is the Hardy–Littlewood maximal operator and we denote by φ_s the translate function $\varphi_s(t)=\varphi(t-s)$, by using the translation properties of Lebesgue measure and the fact that $|t|^\alpha$ is a A_p-weight, we have

$$\int_{\mathbb{R}}|M\varphi(t)|^p(|x_1|^\alpha+|t_0+t|^\alpha)dt$$

$$\begin{aligned}
&= \int_{\mathbb{R}} |M\varphi(t)|^p |x_1|^\alpha dt + \int_{\mathbb{R}} |M\varphi(t)|^p |t_0 + t|^\alpha dt \\
&= |x_1|^\alpha \int_{\mathbb{R}} |M\varphi(t)|^p dt + \int_{\mathbb{R}} |M\varphi_{t_0}(t)|^p |t|^\alpha dt \\
&\leq |x_1|^\alpha C_p \int_{\mathbb{R}} |\varphi(t)|^p dt + A_p(|t|^\alpha) \int_{\mathbb{R}} |\varphi_{t_0}(t)|^p |t|^\alpha dt \\
&\leq (C_p + A_p(|t|^\alpha)) \int_{\mathbb{R}} |\varphi(t)|^p (|x_1|^\alpha + |t_0 + t|^\alpha) dt.
\end{aligned}$$

It follows that $|x_1|^\alpha + |t_0 + t|^\alpha$, and hence $|x + ty'|^\alpha$, is an A_p-weight with an A_p-constant on $\mathbb{R}$ independent of x and y'. ∎

Proof of Theorem B: We only give the proof in the case of the oscillation operator; we leave to the reader the details for the variation operator.

Throughout this proof we shall denote by Δ_j the kernels of the operators $R_j = \partial_{x_j}(-\Delta)^{-1/2}$. We consider the Hilbert transform on $\mathbb{R}$ given by the kernel $k(t) = t^{-1}$. Therefore, by using Lemma 3.4 we have

$$K^j_{\varepsilon,\varepsilon'} f(x) = \frac{2n^{1/2}}{\omega_{n-1}} \int_{\{z \in \mathbb{R}^n : \varepsilon < |z| < \varepsilon'\}} \frac{1}{|z|} f(x - z) \frac{z_j}{|z|^n} dz,$$

and so $K^j_{\varepsilon,\varepsilon'} f(x) = -\kappa_n \Delta_{j,\varepsilon,\varepsilon'} * f(x)$, where $\kappa_n = n^{1/2}\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})/2\Gamma(\frac{n+1}{2})$. Stirling's formula gives $|\kappa_n| \sim C$ and therefore the theorem follows from Corollary 3.7. ∎

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References

[AC] P. Auscher and M. J. Carro, *Transference for radial multipliers and dimension free estimates,* Transactions of the American Mathematical Society **342** (1994), 575–593.

[CJRW] J. T. Campbell, R. L. Jones, K. Reinhold and M. Wierdl, *Oscillation and variation for the Hilbert transform,* Duke Mathematical Journal **105** (2000), 59–83.

[CJRW2] J. T. Campbell, R. L. Jones, K. Reinhold and M. Wierdl, *Oscillation and variation for singular integrals in higher dimensions,* Transactions of the American Mathematical Society **355** (2003), 2115–2137.

[DR] J. Duoandikoetxea and J. L. Rubio de Francia, *Estimations indépendantes de la dimension pour les transformées de Riesz,* Comptes Rendus de l'Académie des Sciences, Paris, Série I **300** (1985), 193–196.

[GT] A. T. Gillespie and J. L. Torrea, *Weighted ergodic theory and dimension free estimates,* The Quarterly Journal of Mathematics. Oxford. Second Series **54** (2003), 257–280.

[JKRW] R. L. Jones, R. Kaufman, J. M. Rosenblatt and M. Wierdl, *Oscillation in ergodic theory,* Ergodic Theory and Dynamical Systems **18** (1998), 889–935.

[MT] F. J. Martín-Reyes and A. de la Torre, *The dominated ergodic estimate for mean bounded, invertible, positive operators,* Proceedings of the American Mathematical Society **104** (1988), 69–75.

[Mu] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function,* Transactions of the American Mathematical Society **165** (1972), 207–226.

[P] G. Pisier, *Riesz transforms: a simpler analytic proof of P.A. Meyer's inequality,* Lecture Notes in Mathematics, Vol. 1321, Springer-Verlag, Berlin, 1988, pp. 485–501.

[R] J. L. Rubio de Francia, *Tranference principles for radial multipliers,* Duke Mathematical Journal **58** (1989), 1–18.

[RRT] J. L. Rubio de Francia, F. J. Ruiz and J. L. Torrea, *Calderón–Zygmund theory for operator-valued kernels,* Advances in Mathematics **62** (1986), 7–48.

[S1] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions,* Princeton University Press, Princeton, N.J., 1970.

[S2] E. M. Stein, *Topics in Harmonic Analysis related to the Littlewood–Paley Theory,* Annals of Mathematics Studies, Princeton University Press, Princeton, N.J., 1970.

[S3] E. M. Stein, *Some results in harmonic analysis in* $\mathbb{R}^n$, *for* $n \to \infty$, Bulletin of the American Mathematical Society (N.S.) **9** (1983), 71–73.